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Pathwise Smoothing of Markov Processes with Noisy Observations

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I. INTRODUCTION

Since the advent of the Feynman–Kac formula [8, 9, 17], the connections between function space integrals arising in probability theory and partial differential equations has been a field of active research. Recently, such connections have found applications in stochastic filtering theory [2, 12–14]. A prototypical estimation problem for random processes can be described as follows: Given a signal process $\{X_t\}_{t\in[0,T]}$ and a related observation process $\{Y_t\}_{t\in[0,T]}$, one wants to estimate $E(X_t/Y_s,\ s\in[0,T])$ for some $t\in[0,T]$. Depending on whether $t\leqslant T$, $t=T$, or $t\geqslant T$, the problem is referred to as a smoothing, filtering, or prediction problem. For $\{X_t\}_{t\in[0,T]}$ a diffusion process, and a signal plus white noise model for $\{Y_t\}_{t\in[0,T]}$, Davis [2] has given a *pathwise* formulation of the filtering problem. This formulation involves expressing the filtered estimate as a function space integral that does not involve stochastic integration with respect to $\{Y_t\}_{t\in[0,T]}$. Using the theory of multiplicative functionals of Markov processes, the problem is then converted to an equivalent problem of solving a deterministic p.d.e. in which the observation process acts like a parameter. This formulation has an advantage over conventional formulations in terms of being robust with respect to errors in the modelling of observation noise, in a precise sense. For a detailed discussion of this aspect, the reader is referred to the original paper of Davis [2].

In this paper, we have given a similar formulation of the smoothing problem. The claim that this formulation is robust with respect to the errors in modelling the observation noise can be justified the same way as in [2], hence the arguments are not repeated. The techniques used here are quite different from those used in [2] and are more in the spirit of [5, 6] (see also [10, pp. 50–52]). For the special case of $t=T$, this approach provides an alternative derivation for some of the results in [2].

The smoothing problem is much more difficult than the filtering problem

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and has received comparatively less attention in literature (see, e.g., [11, 14]). Heuristically, the difficulty can be explained as follows. Unlike the filtering and prediction problems, smoothing takes into account both the past observations $\{Y_s\}_{s\in[0,t]}$ and the future observations $\{Y_s\}_{s\in[t,T]}$. If the signal is Markov and independent of the observation noise, the past and future observations are independent conditioned on X_t, Y_t. Since a time-reversed Markov process is Markov, they can be expected to play a symmetric but independent role. Thus the smoothing problem is essentially a compounding of two filtering-type problems placed back-to-back. This intuition is confirmed by our results. Another source of difficulty stems from the fact that, unlike filtering, there are two independent time points, t and T, with respect to which one may want to update the estimate.

The motivation behind *pathwise* p.d.e. formulations also comes from the fact that they provide a framework for developing approximate computational schemes. For filtering, this is done in [4]. It is hoped that the insight gained from the present work will provide the impetus for corresponding development in smoothing.

II. Preliminaries

Let (Ω, f, P) be a probability triple with $\{f_t\}_{t\in[0,T]}$, $T < \infty$, an increasing right-continuous family of complete sub-σ-fields of f. Let $\{W_t\}_{t\in[0,T]}$ be an (f_t, P)-Wiener process. Consider the stochastic differential equation

$$dX_t = m(X_t, t)\,dt + \sigma(X_t, t)\,dW_t, \qquad X_0 = 0 \qquad \text{a.s.} \quad t \in [0, T], \tag{2.1}$$

where m: $R \times [0, T] \to R$, $\sigma : R \times [0, T] \to R$ are measurable maps.

Before proceeding further, we introduce some useful notions.

DEFINITION 2.1. A measurable map q: $R \times [0, T] \to R$ is said to be of class 0 if there exists a positive constant K such that

$$|q(x, t)| \leqslant K\sqrt{1 + x^2} \tag{2.2a}$$

$$|q(x, t) - q(y, t)| \leqslant K\,|x - y| \tag{2.2b}$$

for $t \in [0, T]$, $x, y \in R$. If in addition, $q'(x, t) = (\partial/\partial x)\, q(x, t)$ exists, it is of class 1. If $q''(x, t) = (\partial^2/\partial x^2)\, q(x, t)$ also exists, it is of class 2. Finally, if $\dot{q}(x, t) = (\partial/\partial t)\, q(x, t)$ exists as well, it is said to be of class 3.

Coming back to (2.1), we make the following assumptions:

A1. m is of class 1 and σ is of class 2.

A2. There exists a positive number σ_0 such that

$$\sigma(x, t) \geqslant \sigma_0 > 0 \qquad \forall t \in [0, T], \quad x \in R.$$

Under these conditions, it is well known [11, 16] that (2.1) has a unique strong solution which is strong Markov. Also, $\{X_t\}_{t \in [0, T]}$ has a.s. continuous sample paths and induces a probability measure P_x on $C[0, T]$ (the Banach space of continuous functions $[0, T] \to R$ with the supremum norm). Let $E_x(\cdot)$ denote the expectation with respect to P_x. Consider the function space integral

$$E_x \left[f(X_T, T) \exp \left\{ \int_0^T \alpha(X_s, s)\, dX_s - \int_0^T \beta(X_s, s)\, ds \right\} \right]. \tag{2.3}$$

Here, f, α, β are suitable maps such that expectation (2.3) is well defined.

Remarks. Suppose α is of class 3. Let

$$h(x, t) = \int_0^x \alpha(y, t)\, dy.$$

By the Ito differentiation rule,

$$\int_0^T \alpha(X_s, s)\, dX_s = h(X_T, T) - \int_0^T \left(\dot{h}(X_s, s) + \frac{\alpha'(X_s, s)\, \sigma^2(X_s, s)}{2} \right) ds,$$

which, on substitution, simplifies (2.3) by removing the Ito integral term. For the applications we have in mind, however, one cannot assume that $(\partial/\partial t)\, \alpha(x, t)$ exists and hence α cannot be assumed to be of class 3.

We make the following assumptions:

A3. $P_x(\int_0^T \sigma^{-2}(X_s, s)\, m^2(X_s, s)\, ds < \infty) = 1$, $\quad P_x(\int_0^T \sigma^2(X_s, s)\, \alpha^2(X_s, s)\, ds < \infty) = 1$.

A4. $E_w(\exp\{-\int_0^T \alpha(W_s, s)\, dW_s - \frac{1}{2} \int_0^T \alpha^2(W_s, s)\, ds\}) = 1$, where $E_w(\cdot)$ denotes the expectation with respect to the Wiener measure on $C[0, T]$.

A5. $m_2 : R \times [0, T] \to R$ defined by

$$m_2(x, t) = m(x, t) + \alpha(x, t)\, \sigma^2(x, t) \tag{2.4}$$

is of class 1.

Remarks. Sufficient conditions for A4 to hold are given in [1, 7].

It follows then that the stochastic differential equation

$$d\eta_t = m_2(\eta_t, t)\,dt + \sigma(\eta_t, t)\,dW_t, \tag{2.5}$$

has a unique strong solution which is strong Markov. In addition, suppose that

A6. $\{\eta_t\}_{t\in[0,t]}$ has a transition probability density $p_\eta(x, t/x_0, t_0)$, $t > t_0$, which is twice differentiable in x and once in t.

One can give a stronger condition that will ensure that A6 holds ([16, pp. 173]).

A6′. m_2', σ', σ'' are of class 0.

Under A6, $p_\eta(n, t/x_0, t_0)$ satisfies the forward Kolmogorov equation [16].

$$\begin{aligned}\frac{\partial}{\partial t}\, p_\eta(x, t/x_0, t_0) &= \frac{1}{2}\frac{\partial^2}{\partial x^2}\,[\sigma^2(x, t)\, p_\eta(x, t/x_0, t_0)]\\ &\quad - \frac{\partial}{\partial x}\,[m_2(x, t)\, p_\eta(x, t/x_0, t_0)]\end{aligned} \tag{2.6}$$

for $t > t_0$, t, $t_0 \in [0, T]$, with

$$p_\eta(x, t_0/x_0, t_0) = \partial(x - x_0). \tag{2.7}$$

Also, $\{\eta_t\}_{t\in[0,T]}$ has a.s. continuous sample paths and induces a probability measure P_η on $C[0, T]$. P_η, P_x are mutually absolutely continuous with the Radon–Nikodym derivative ([11, pp. 273–275])

$$\frac{dP_x}{dP_\eta} = \exp\left\{-\int_0^T \alpha(\eta_s, s)\,d\eta_s - \frac{1}{2}\int_0^T \sigma^{-2}(\eta_s, s)(m^2(\eta_s, s) - m_2^2(\eta_s, s))\,ds\right\}. \tag{2.8}$$

Thus

$$\begin{aligned}E_x&\left[f(X_T, T)\exp\left\{\int_0^T \alpha(X_s, s)\,dX_s - \int_0^T \beta(X_s, s)\,ds\right\}\right]\\ &= E_\eta\left[f(\eta_T, T)\exp\left\{-\int_0^T V(\eta_s, s)\,ds\right\}\right]\end{aligned} \tag{2.9}$$

with $E_\eta(\cdot)$ denoting the expectation with respect to P_η and

$$V(x, s) = \beta(x, s) + \tfrac{1}{2}\sigma^{-2}(x, s)(m^2(x, s) - m_2^2(x, s)). \tag{2.10}$$

We next derive a p.d.e. associated with the right-hand side of (2.9).

III. Function Space Integrals and P.D.E.S.

On a few occasions in this section, we shall implicitly use the Fubini–Tonelli theorem and differentation under the integral sign (cf. the last statement of the section). It is assumed that they are permissible wherever used. Define u: $[0, T] \times R \to R$ by

$$u(t, \xi) = \int_{-\infty}^{\infty} e^{iy\xi} E_\eta \left[\exp\left(-\int_0^t V(\eta_s, s)\, ds\right) e^{-iy\eta_t}\right] dy.$$

Lemma 3.1. *For a measurable $f(x, t)$: $R \times [0, T] \to R$ which is locally integrable in x for fixed t,*

$$E_\eta \left[f(\eta_t, t) \exp\left(-\int_0^t V(\eta_s, s)\, ds\right)\right] = \int_{-\infty}^{\infty} f(\xi, t)\, u(t, \xi)\, d\xi.$$

Proof.

$$\int_{-\infty}^{\infty} f(\xi, t)\, u(t, \xi)\, d\xi$$

$$= \int_{-\infty}^{\infty} f(\xi, t) \int_{-\infty}^{\infty} e^{iy\xi} E_\eta \left(\exp\left(-\int_0^t V(\eta_s, s)\, ds\right) e^{-iy\eta_t}\right) dy\, d\xi$$

$$= E_\eta \left[\exp\left(-\int_0^t V(\eta_s, s)\, ds\right) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, t)\, e^{iy\xi - iy\eta_t}\, dy\, d\xi\right]$$

$$= E_\eta \left[f(\eta_t, t) \exp\left(- V(\eta_s, s)\, ds\right)\right]$$

by the Fourier inversion formula. Q.E.D.

Lemma 3.2. $\int_{-\infty}^{\infty} e^{iy\xi} E_\eta[e^{-iy\eta_t}/f_{t_0}^\eta]\, dy = p_\eta(\xi, t/\eta_{t_0}, t_0)$, *where* $t > t_0$; t, $t_0 \in [0, T]$ *and* $f_{t_0}^\eta$ *is the σ-field generated by* $\{\eta_s, s \in [0, t_0]\}$.

Proof.

$$\int_{-\infty}^{\infty} e^{iy\xi} E_\eta(e^{-iy\eta_t}/f_{t_0}^\eta)\, dy = \int_{-\infty}^{\infty} e^{iy\xi} \int_{-\infty}^{\infty} e^{-iyx}\, p_\eta(x, t/\eta_{t_0}, t_0)\, dx\, dy.$$

Remarks. The above results can also be obtained by using Donsker's delta function ([10, pp. 50–51]). The arguments will be essentially identical.

Theorem 3.1. *$u(t, \xi)$ is a solution of*

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial \xi^2} (\sigma^2(\xi, t)\, u(t, \xi)) - \frac{\partial}{\partial \xi} (m_2(\xi, t)\, u(t, \xi)) - V(\xi, t)\, u(t, \xi)$$

with

$$u(t, \xi) \to 0 \quad \text{as} \quad \xi \to \pm\infty, \qquad u(t, \xi) \to \delta(\xi) \quad \text{as} \quad t \to 0.$$

Proof. Note that

$$\exp\left(-\int_0^t V(\eta_s, s)\, ds\right) = 1 - \int_0^t V(\eta_\tau, \tau) \exp\left(-\int_0^\tau V(\eta_s, s)\, ds\right) d\tau.$$

Hence

$$u(t, \xi) = \int_{-\infty}^{\infty} e^{iy\xi} E_\eta(e^{-iy\eta_t})\, dy - \int_{-\infty}^{\infty} e^{iy\xi} \int_0^t E_\eta \Bigg(V(\eta_\tau, \tau) \times \exp\left(-\int_0^\tau V(\eta_s, s)\, ds\right) e^{-iy\eta_t}\Bigg)\, d\tau\, dy. \tag{3.1}$$

Now, by Lemma 3.2,

$$\int_{-\infty}^{\infty} e^{iy\xi} E_\eta(e^{-iy\eta_t})\, dy = p(\xi, t/0, 0)$$

By Lemmas 3.1 and 3.2,

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{iy\xi} \int_0^t E_\eta \left[V(\eta_\tau, \tau) \exp\left(-\int_0^\tau V(\eta_s, s)\, ds\right) e^{-iy\eta_t}\right] d\tau\, dy \\
&= \int_{-\infty}^{\infty} e^{iy\xi} \int_0^t E_\eta \left[V(\eta_\tau, \tau) \exp\left(-\int_0^\tau V(\eta_s, s)\, ds\right) E_\eta(e^{-iy\eta_t}/f_\tau^\eta)\right] d\tau\, dy \\
&= \int_0^t E_\eta \left[V(\eta_\tau, \tau) \exp\left(-\int_0^\tau V(\eta_s, s)\, ds\right) \int_{-\infty}^{\infty} e^{iy\xi} E_\eta(e^{-iy\eta_t}/f_\tau^\eta)\, dy\right] d\tau \\
&= \int_0^t E_\eta \left[V(\eta_\tau, \tau) \exp\left(-\int_0^\tau V(\eta_s, s)\, ds\right) p(\xi, t/\eta_\tau, \tau)\right] d\tau \\
&= \int_0^t \int_{-\infty}^{\infty} V(z, \tau)\, u(\tau, z)\, p(\xi, t/z, \tau)\, dz\, d\tau.
\end{aligned}$$

Then (3.1) becomes

$$u(t, \xi) = p(\xi, t/0, 0) - \int_0^t \int_{-\infty}^{\infty} V(z, \tau)\, u(\tau, z)\, p(\xi, t/z, \tau)\, dz\, d\tau. \tag{3.2}$$

Using (2.6), (2.7), one can verify by direct substitution that the desired results hold. Q.E.D.

IV. Formulation of the Smoothing Problem

Consider (2.1) in conjunction with an observation equation

$$dY_t = h(X_t, t)\,dt + r(t)\,dW_{2t}, \qquad Y_0 = 0 \qquad \text{a.s.}, \quad t \in [0, T], \tag{4.1}$$

where $\{W_{2t}\}_{t\in[0,T]}$ is an (f_t, P)-Wiener process independent of $\{W_t\}_{t\in[0,T]}$. Assume:

A7. h is of class 3 and $h'\sigma^2$ is of class 1.

A8. $r(t)$ is differentiable and uniformly bounded away from 0 with a bounded derivative $\dot{r}(t)$.

It is clear that $\{Y_t\}_{t\in[0,T]}$ has a.s. continuous sample paths and hence the process $\{(X_t, Y_t)\}_{t\in[0,T]}$ induces a probability measure P_1 on $C[0,T]\times C[0,T]$. Define a probability measure P_0 on $C[0,T]\times C[0,T]$ by

$$\Lambda = \frac{dP_1}{dP_0} = \exp\left\{\int_0^T \frac{h(X_s, s)}{r^2(s)}\,dY_s - \frac{1}{2}\int_0^T \frac{h^2(x_s, s)}{r^2(s)}\,ds\right\}.$$

Then (by the Cameron–Martin–Girsanov theorem [15, 16]) under P_0, $\{X_t\}_{t\in[0,T]}$, $\{Y_t\}_{t\in[0,T]}$ are independent, $\{X_t\}_{t\in[0,T]}$ has the same distribution as under P_1 and there is a Brownian motion $\{W_{2t}\}$ independent of $\{X_t\}$ such that the distribution of $\{Y_t\}_{t\in[0,T]}$ conforms to the dynamics

$$dY_t = r(t)\,dW_{2t}, \qquad Y_0 = 0 \qquad \text{a.s.}, \quad t \in [0, T].$$

See [15, 16] for further details.

By a variant of the Ito differentiation rule (see [3]),

$$\int_0^t \frac{h(X_s, s)}{r^2(s)}\,dY_s = \frac{Y_t h(X_t, t)}{r^2(t)} - \int_0^t \frac{Y_s h'(X_s, s)}{r^2(s)}\,dX_s - \int_0^t Y_s\left[\frac{h''(X_s, s)\,\sigma^2(X_s, s)}{2r^2(s)} + \frac{\dot{h}(X_s, s)}{r^2(s)} - \frac{2h(X_s, s)\,\dot{r}(s)}{r^3(s)}\right] ds.$$

Letting

$$g_1(x, t) = h(x, t)/r^2(t), \qquad g_2(x, t) = -h'(x, t)/r^2(t),$$

$$g_3(y, x, t) = \frac{yh''(x, t)\,\sigma^2(x, t) + h^2(x, t) + 2y\dot{h}(x, t)}{2r^2(t)} - \frac{2h(x, t)\,\dot{r}(t)}{r^3(t)}\,y$$

for $x, y \in R$, $t \in [0, T]$, we have

$$\Lambda = \exp\left\{Y_T\, g_1(X_T, T) + \int_0^T Y_s\, g_2(X_s, s)\,dX_s - \int_0^T g_3(Y_s, X_s, s)\,ds\right\}.$$

Let $E_1(\cdot)$, $E_0(\cdot)$ denote the expectations with respect to P_1, P_0, respectively. The (least-square) smoothing problem is to calculate, for $t \in [0, T]$ and some bounded continuous Φ: $R \to R$, the conditional expectation

$$\hat{\Phi} = E_1(\Phi(X_t)/f^{\mathscr{Y}}),$$

where $f^{\mathscr{Y}}$ is the σ-field generated by $\{Y_s, s \in [0, T]\}$. (For $t = T$, this coincides with the filtering problem.) It is easily seen that [3, 16]

$$\hat{\Phi} = E_0(\Phi(X_t)\Lambda/f^{\mathscr{Y}})/E_0(\Lambda/f^{\mathscr{Y}}). \tag{4.2}$$

The denominator is simply a normalising factor and thus we confine ourselves to evaluating

$$\tilde{\Phi} = E_0[\Phi(X_t)\Lambda/f^{\mathscr{Y}}_{\mathscr{T}}]. \tag{4.3}$$

(Note that if $\Phi \equiv 1$, (4.3) yields the denominator of (4.2)). Using the independence of $\{X_t\}_{t \in [0,T]}$ and $\{Y_t\}_{t \in [0,T]}$ under P_0, we can rewrite (4.3) as

$$\tilde{\Phi} = E_{XY}[\Phi(X_t)\Lambda],$$

where $E_{XY}[\cdot]$ indicates the expectation taken with respect to the distribution of $\{X_t\}_{t \in [0,T]}$, which is the same under P_1 and P_0, keeping $\{Y_t\}_{t \in [0,T]}$ fixed as a parameter. Let $\Lambda = \Lambda_1 \Lambda_2$ with

$$\Lambda_1 = \exp\left\{\int_0^t Y_s g_2(X_s, s)\, dX_s - \int_0^t g_3(Y_s, X_s, s)\, ds\right\}$$

$$\Lambda_2 = \exp\left\{Y_T g_1(X_T, T) + \int_t^T Y_s g_2(X_s, s)\, dX_s - \int_t^T g_3(Y_s, X_s, s)\, ds\right\}.$$

Then

$$\tilde{\Phi} = E_{XY}[\Phi(X_t)\Lambda_1\Lambda_2] = E_{XY}[\Phi(X_t)\Lambda_1 E_{XY}(\Lambda_2/X_t)],$$

where the Markov property of $\{X_t\}_{t \in [0,T]}$ is used. Letting $\Psi(x, t) = E_{XY}(\Lambda_2/X_t = x)$, we have

$$\tilde{\Phi} = E_{XY}[\Phi(X_t)\Lambda_1 \Psi(X_t, t)]. \tag{4.4}$$

V. P.D.E.S. Associated with the Smoothing Problem

Define

$$\bar{X}_{sx} = X_{s+t} - x, \qquad \tilde{m}(z, s) = m(z, s) + Y_s g_2(z, s)\, \sigma^2(z, s),$$

$$\tilde{v}(z, s) = g_3(Y_s, z, s) + \tfrac{1}{2}\sigma^{-2}(z, s)[m^2(z, s) - \tilde{m}^2(z, s)].$$

THEOREM 5.1. *$\Psi(x, t)$, $x \in \mathbb{R}$, is given by*

$$\Psi(x, t) = \int_{-\infty}^{\infty} \exp[Y_T g_1(y, T)]\, a(t, x; T, y)\, dy,$$

where $a(t, x; s, y)$, $s \in [t, T]$; $x, y \in \mathbb{R}$, satisfies

$$\frac{\partial}{\partial s} a(t, x; s, y) = \frac{1}{2} \frac{\partial^2}{\partial y^2} [\sigma^2(y, s)\, a(t, x; s, y)] - \frac{\partial}{\partial y} [\tilde{m}(y, s)\, a(t, x; s, y)] - \tilde{v}(y, s)\, a(t, x; s, y)$$

with

$$a(t, x; s, y) \to 0 \quad \text{as} \quad y \to \pm\infty, \qquad a(t, x; s, y) \to \delta(y - x) \quad \text{as} \quad s \to t.$$

Proof. Note that $\{\bar{X}_{sx}\}_{s \in [0, T-t]}$ satisfies

$$\bar{X}_{sx} - \bar{X}_{0x} = \int_0^s m(\bar{X}_{ux} + x, t + u)\, dy + \int_0^s \sigma(\bar{X}_{ux} + x, t + u)\, dW_u. \quad (5.1)$$

Also

$$\begin{aligned}
\Psi(x, t) &= E_{XY}(\Lambda_2 / X_t = x) \\
&= E_{XY} \left[\exp \left\{ Y_T g_1(X_T, T) + \int_t^T Y_s g_2(X_s, s)\, dX_s \right.\right. \\
&\quad \left.\left. - \int_t^T g_3(Y_s, X_s, s)\, ds \right\} \Big/ X_t = x \right] \\
&= \bar{E}_X \left[\exp \left\{ Y_T g_1(\bar{X}_{(T-t)x} + x, T) + \int_0^{T-t} Y_{t+s} g_2(\bar{X}_{sx} + x, t + s)\, d\bar{X}_{sx} \right.\right. \\
&\quad \left.\left. - \int_0^{T-t} g_3(Y_{t+s}, \bar{X}_{sx} + x, t + s)\, ds \right\} \Big/ \bar{X}_{0x} = 0 \right],
\end{aligned}$$

where $\bar{E}_x(\cdot)$ denotes the expectation with respect to the distribution of $\{\bar{X}_{sx}\}_{s \in [0, T-t]}$ satisfying (5.1) with $\bar{X}_{0x} = 0$, treating $\{Y_s\}_{s \in [t, T]}$ as a fixed parameter. From Theorem 3.1, it follows that,

$$\Psi(x, t) = \int_{-\infty}^{\infty} \exp[Y_R g_1(x + y, T)]\, c(t, x; T - t, y)\, dy,$$

where $c(t, x; s, y)$, $s \in [0, T-t]$, $y \in \mathbb{R}$, satisfies

$$\frac{\partial}{\partial s} c(t, x; s, y) = \frac{1}{2} \frac{\partial^2}{\partial y^2} [\sigma^2(x+y, s+t)\, c(t, x; s, y)]$$
$$- \frac{\partial}{\partial y} [\tilde{m}(x+y, s+t)\, c(t, x; s, y)] - \tilde{v}(x+y, s+t)\, c(t, x, s, y)$$

with

$$c(t, x; s, y) \to 0 \quad \text{as } y \to \pm\infty, \quad c(t, x; s, y) \to \delta(y) \quad \text{as } s \to 0.$$

Let $a(t, x; s, y) = c(t, x; s-t, y-x)$ for $s \in [t, T]$, $y \in \mathbb{R}$ to get the desired result. Q.E.D.

The following corollary is immediate:

COROLLARY 5.1. *$\tilde{\Phi} = \int_{-\infty}^{\infty} \Phi(y)\, \Psi(y, t)\, b(t, y)\, dy$, where $b(s, y)$, $s \in [0, t]$, $y \in \mathbb{R}$ satisfies*

$$\frac{\partial}{\partial s} b(s, y) = \frac{1}{2} \frac{\partial}{\partial y^2} [\sigma^2(y, s)\, b(s, y)] - \frac{\partial}{\partial y} [\tilde{m}(y, s)\, b(s, y)]$$
$$- \tilde{v}(y, s)\, b(s, y)$$

with

$$b(s, y) \to 0 \quad \text{as} \quad y \to \pm\infty, \quad b(s, y) \to \delta(y) \quad \text{as} \quad s \to 0.$$

This gives the following overall expression for the smoothed estimate

$$\tilde{\Phi} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(x) \exp[Y_T\, g_1(z, T)]\, a(t, x; T, z)\, b(t, x)\, dz\, dx.$$

VI. CONCLUSIONS

The smoothing problem for Markov processes with noisy observations has been given an alternative formulation in terms of certain p.d.e.s. associated with it. These p.d.e.s. are deterministic, with the observation process appearing as a parameter. This formulation has the advantage of being robust with respect to the errors in modelling observation noise [2].

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